a geometric constraint, then a slight modification of the procedure presented above ensures the stable encounter of all approximation motions $x_\Delta\,[t]$ with the ε-neighborhood of set M^* by the instant ϑ denoting an arbitrarily large quantity.

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UDC 62-50

SYNTHESIS OF TIME-OPTIMAL CONTROL OF A THIRD-ORDER OBJECT WITH A PHASE CONSTRAINT

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We examine a problem, arising in engineering practice, of the time-optimal control of a third-order linear object with a constraint on the control and on the phase coordinate . The synthesis of the control is described.

1. Statement of the problem. The following problem arises in the combined operation of two measuring devices tracking a moving object, each of which can track only in a certain part (action zone) of the space of measurements. From the information on the object obtained by the measuring device of the leaving zone, organize maximally quickly the tracking by the measurment device of the entering zone. For a number of measuring devices the dynamics of the tracking organization process can be described by a system of linear differential equations with constant coefficients and with constraints on the control u and on the phase coordinate (the action zone of the measuring device)

$$\frac{dx_1}{dt} = \Omega - x_3, \quad \frac{dx_2}{dt} = x_3, \quad \frac{dx_3}{dt} = -\frac{1}{T} x_3 + u \tag{1.1}$$

$$U \leqslant u < U, \quad x_{2_0}{}^2 - x_2{}^2 \leqslant 0 \tag{1.2}$$

The control $u^\circ(t)$, ensuring the satisfaction of condition (1.2) and translating the object (1.1) in a minimal time t_f from a specified initial point $x_1(0) = x_1{}^*$, $x_2(0) = x_2{}^*$, $x_3(0) = x^*{}_3$ (the initial position of the measuring device) to a specified final point $x_1(t_f) = 0$, $x_3(t_f) = \Omega$ (the condition for tracking to commence), is assumed to be optimal. In the problem being investigated it is of interest to synthesize the optimal control ensuring the system's operation for each fixed set of parameters T, U, x_{20} (from the sets $T > 0$, $U > 0$, $x_{20} > 0$) under arbitrary coordinates of the initial point and arbitrary values, constant in time, of the parameter Ω.

2. Properties of the optimal control. From the maximum principle [1] it follows that if

$$H = \psi_1(\Omega - x_3) + \psi_2 x_3 + \psi_3\left(-\frac{1}{T}x_3 + u\right) - \frac{d\mu}{dt}(x_2^2 - x_{20}^2)$$

$$l = \alpha_y t_f + C_1[x_1(0) - x_1^*] + C_2[x_2(0) - x_2^*] + C_3[x_3(0) - x_3^*] + C_4 x_1(t_f) + C_5[x_3(t_f) - \Omega]$$

$$\alpha_y \geqslant 0, \quad \frac{d\mu}{dt} \geqslant 0, \quad \frac{d\mu}{dt}[x_{20}^2 - (x_2^\circ(t))^2] = 0$$

for arbitrary constants C_i, $i = 1, 2 \ldots 5$, then the adjoint variables satisfy the equations

$$-\frac{d\psi_1}{dt} = 0, \quad \psi_1(0) = C_1, \quad \psi_1(t_f) = -C_4$$

$$-\frac{d\psi_2}{dt} = -2x_2(t)\frac{d\mu}{dt}, \quad \psi_2(0) = C_2, \quad \psi_2(t_f) = 0$$

$$-\frac{d\psi_3}{dt} = -\psi_1 + \psi_2 - \frac{1}{T}\psi_3, \quad \psi_3(0) = C_3, \quad \psi_3(t_f) = C_5$$

$$-\frac{d\psi_t}{dt} = 0, \quad \psi_t(0) = 0, \quad \psi_t(t_f) = -\alpha_y$$

Hence

$$\psi_1(t) = \psi_1(0), \quad C_1 = -C_4 \tag{2.1}$$

$$\psi_2(t) = \psi_2(0) + \int_0^t 2x_2^\circ(\tau)\frac{d\mu}{d\tau}d\tau$$

$$\psi_3(t) = [\psi_3(0) + (\psi_1(0) - \psi_2(0))T]e^{t/T} - [\psi_1(0) - \psi_2(0)]T - e^{t/T}\int_0^T e^{-\tau/T}\int_0^\tau 2x_2^\circ(\zeta)\frac{d\mu}{d\zeta}d\zeta\, d\tau$$

$$\psi_t(t) = \text{const}, \quad \alpha_y = 0$$

The optimal control is determined by the expression $u^\circ(t) = U\,\mathrm{sign}\,\psi_3(t)$.

It is convenient to investigate the problem being examined, which belongs to the type of problems studied in [2], in the coordinates

$$y_1 = x_3 - x_1 / T, \quad y_2 = x_3 - Tu^\circ, \quad y_3 = x_2 + Tx_3$$

In the new variables we obtain

$$y_1(0) = x_3(0) - x_1(0) / T, \quad y_3(0) = x_2(0) + Tx_3(0) \tag{2.2}$$

$$y_2^\mp = x_3(0) \pm TU \quad \text{for} \quad u^\circ = \mp U$$

$$\frac{dy_1}{dt} = U - \frac{1}{T}\Omega, \quad \frac{dy_2}{dt} = -\frac{y_2}{T}, \quad \frac{dy_3}{dt} = TU \tag{2.3}$$

$$y_2 = C\exp[-y_1(TU - \Omega)^{-1}], \quad y_3 = C\exp[-y_3(T^2U)^{-1}] \tag{2.4}$$

$$(u = \text{const})$$

$$(y_3 + x_{20})T^{-1} \pm TU \geqslant y_2 \geqslant (y_3 - x_{20})T^{-1} \pm TU \tag{2.5}$$

for the two possible values $u^\circ = \pm U$. Here (2. 2) gives the values of the coordinates of the initial point, (2. 3) is a system of differential equations (with due regard to the substitution made) and (2. 4) gives the equations of the phase trajectories. We partition the domains of parameters in the following manner:

$$\text{a) } \Omega \geqslant 0, \ \Omega < 0, \text{ (b) } TU - |\Omega| \geqslant 0, \ TU - |\Omega| < 0,$$
$$\text{c) } x_{20} \geqslant T^2 U, \ x_{20} < T^2 U.$$

The analysis in Sects. 2 and 3 is carried out for the combination $\Omega > 0$, $TU - \Omega > 0$, $x_{20} > T^2U$.

2. 1. If the optimal control is such that $d\mu / dt \equiv 0$, then it coincides with the control for the corresponding time-optimal problem without phase constraints. From (2. 1) it follows that the optimal control is a piecewise-constant function having no more than one switching. A complete representation of the system's optimal motion can be obtained from the phase trajectories in the plane $\{y_1, y_2\}$.

Figure 1 shows individual phase trajectories (2. 4) corresponding to the controls $\pm U$. We note that a change of sign of the control leads to a jump of the phase point by the amount $\pm 2TU$ with respect to the coordinate y_2. We can hit the final point $y_1 = \Omega$, $y_2 = \Omega \pm TU$ by moving along a trajectory with the control $u = \mp U$ passing through it (the solid parts of curves $f_1(y_1)$ and $f_2(y_2)$). If the phase point corresponding to the initial state of the system is located on such a trajectory, then the system is translated to the final state without a change in sign of the control. Otherwise, the control's sign has to be changed at such a point of the plane $\{y_1, y_2\}$ that the jump from it with respect to coordinate y_2 of magnitude $\pm 2TU$ ensures falling onto the trajectory determined above. The locus of switching points is shown in Fig. 1 by dashed curves. The functions $y_2 = f_1(y_1)$ and $y_2 = f_2(y_1)$ are determined by the relations

$$f_1: 2TU + (\Omega - TU)\exp\left[\frac{\Omega - y_1}{TU - \Omega}\right], \quad y_1 \leqslant \Omega \tag{2. 6}$$
$$(\Omega + TU)\exp\left[-\frac{\Omega - y_1}{TU + \Omega}\right], \quad y_1 > \Omega, \quad f_2 = f_1 - 2TU$$

2. 2. If the optimal control is such that $d\mu / dt \not\equiv 0$, the phase constraint is an essential one. From Eq. (1. 1) we see that the system can be situated on it ($x_2^2(t) = x_{20}^2$) in the course of some nonzero time interval only when $x_3 = 0$ and $u^\circ = 0$. Such a state is shown on the plane $\{y_2, y_3\}$ by the points $(\pm TU, \pm x_{20})$ and $(0, \pm x_{20})$ subsequently called the rest points Π_1 and Π_2 (Fig. 2). When the system is situated at a rest point ($u = 0$) we have the following pattern of the motion. In the coordinates $\{y_2, y_3\}$ the point remains fixed: $y_2 = 0$ and $y_3 = \pm x_{20}$. In the coordinates $\{y_1, y_2\}$ the point moves in accordance with the equations

$$y_1(t) = y_1(t_b) - \Omega T^{-1}(t - t_b), \quad y_2(t) = 0$$

where t_b denotes the instant at which the trajectory goes out onto the phase constraint.

We note that in the plane $\{y_2, y_3\}$ the control process termination point $x_1(t_f) = 0$, $x_3(t_f) = \Omega$ must lie on the straight line $y_2 = \Omega - TU$ when the process terminates with $u^\circ = +U$ and on the straight line $y_2 = \Omega + TU$ when the process terminates with $u^\circ = -U$. The phase constraints (2. 5) are zones bounded by the parallel lines Γ_1 and Γ_2 (Γ_3 and Γ_4), forbidden to the trajectories with controls $+U$ $(-U)$ (Fig. 2).

When the phase constraint Γ_1 is approached with $u^\circ = +U$ a trajectory exists tangent to Γ_1 at the point

$$\{y_2 = -TU, \quad y_3 = x_{20}\} : h_2 = -TU \exp \quad (x_{20} - y_3)\ (T^2U)^{-1}$$

The tangency occurs at a rest point. The phase trajectory with $u^\circ = +U$ cannot go into the points of boundary Γ_1 with coordinates $y_3 > x_{20}$ but in principle it is possible for the trajectory to go into the points with coordinates $y_3 < x_{20}$. However, in the latter case the phase constraint is violated whether u° preserves or changes its sign. When Γ_4 is approached with $u^\circ = -U$ the corresponding trajectory is tangent to Γ_4 at the point

$$\{y_2 = TU,\ y_3 = -x_{20}\} : h_1 = TU \exp (x_{20} + y_3)\ (T^2U)^{-1}$$

The phase trajectory with $u^\circ = -U$ cannot go into the points of boundary Γ_4 with coordinates $y_3 < -x_{20}$, but in principle it is possible for the trajectory to go into the points with coordinates $y_3 > -x_{20}$. However, in the latter case the phase constraint is violated whether u° preserves or changes its sign.

Let us consider other possibilities of going onto the phase constraints. Going onto constraint Γ_2 with $u^\circ = +U$ is possible at any point. To prevent a violation of the phase constraint at the instant of going out, the control must change sign. The phase constraint is violated when going onto Γ_2' at point with coordinates $y_2 > -x_{20}$, but it is not violated when going into points with $y_2 < -x_{20}$. The integral summand in the expression for $\psi_3(t)$ is positive when going into the point $y_3 = -x_{20}$. Therefore, the trajectory can remain on the phase constraint at this point ($u^\circ = 0$) when the nonintegral summand becomes negative at the instant of going onto the constraint.

Thus, going onto the boundary Γ_2 must always be accompanied by a switching of the control or a stopping of the process. In the first case the successive section of the trajectory with $u^\circ = -U$ is the last one and must arrive at the required final point, i.e. on the line $y_2 = \Omega + TU$ (Fig. 2). After the switching the phase point on the plane $\{y_2, y_3\}$ is located below the optimal line and cannot hit onto it with the control $u^\circ = -U$. Therefore, the required optimal trajectory must not go onto the phase constraint. An exception is the phase trajectory which hits the final point (on the straight line $y_2 = \Omega - TU$ in the plane $\{y_2, y_3\}$) as it goes onto Γ_2.

Analogously, as Γ_2 is approached with $u^\circ = +U$ the phase point can stay on the constraint only at x_{20}. For this it is necessary that $\psi_3(t) = 0$. If the integral summand in the expression for ψ_3 does not equal zero at this point, then it is negative; the nonintegral summand must change sign at the instant of going out. However, in this case, after coming off the constraint $d\mu / dt \equiv 0$ the quantity $\psi_3(t)$ becomes negative, i.e. the phase constraint is violated. Therefore, the coming off is possible only with $u^\circ = -U$, which in its own turn can be implemented only when $d\mu / dt \equiv 0$.

Thus, when Γ_1 is approached with $u^\circ = +U$ the phase trajectories pass the point Π_2 with $d\mu / dt \equiv 0$. Going onto constraint Γ_3 with $u^\circ = -U$ is possible at any point. To prevent a violation of the phase constraint a switching must take place at the instant of going out: a change of control form $u^\circ = -U$ to $u^\circ = +U$. When going onto Γ_3 at points with coordinates $y_3 < x_{20}$ the phase constraint is violated. It is not violated when going into points with $y_3 > x_{20}$. When going onto the point $y_3 = x_{20}$ the integral summand in the expression for $\psi_3(t)$ is negative. Therefore, the trajectory can remain on the phase constraint at this point ($u = 0$) when the nonintegral summand becomes positive at the instant of going onto the constraint.

The going onto the phase boundary Γ_3 must always be accompanied by a switching of the control or a stopping of the process. In the first case the successive section of the trajectory with $u^\circ = +U$ is the last one and must arrive at the required final point, i.e. on the line $y_2 = \Omega - TU$. This is realized when the trajectory with $u^\circ = -U$ goes onto the phase constraint Γ_3, if the point of going out is situated on the segment $[K_1, K_2]$ (Fig. 2). We note that the quantity $d\mu / dt \not\equiv 0$ when going into point K_2. Going into points of Γ_3, located above K_1, leads to the phase point being located above the line $\Omega - TU$ after the switching and not being able to hit onto it with the control $u^\circ = +U$.

Let us investigate the case when the nonintegral summand in the expression for $\psi_3(t)$ equals zero. This is possible when $\psi_3(0) + [\psi_1(0) - \psi_2(0)]\,T = 0$ and $\psi_1(0) - \psi_2(0) = 0$. However, $H = \psi_1(0)(\Omega - x_3) + \psi_2(0)\,x_3$ must be satisfied on the initial segment, i.e. $\psi_1(0) = \psi_2(0) = 0$. Recalling the requirement $\psi_2(t_f) = 0$, we conclude (2.1) that $d\mu / dt \equiv 0$. The case of the vanishing of the nonintegral term leads to the consideration of the degenerate case when $\|\psi_1\| + \|\psi_2\| + \|\psi_3\| + \|d\mu / dt\| = 0$.

2.3. The set of initial states has four characteristic domains S_k, $k = 1, 2, 3, 4$, all of which we could not successfully depict graphically in Figs. 1 and 2. The first domain $\{y_i(0)\} \in S_1$ admits of control without going onto a constraint. For the second domain $\{y_i(0)\} \in S_2$ the initial conditions are: (a) it is impossible to find a control which, in view of its boundedness, could ensure the nonviolation of the constraint (the zone between the straight line Γ_3 and the tangent trajectory $\gamma_1 = h_2 + 2TU$ raised by an amount $2TU$ with respect to the y_2-axis); (b) the magnitude of control U proves to be insufficient for falling into the final state with one switching (the zone between the straight line Γ_3 and the trajectory $\gamma_2 = (\Omega + TU)\exp[-(x_{20} + \Omega T - y_3)](T^2 U)^{-1}$. Analogous zones can be also determined for the boundary Γ_2. Domain S_2 is defined by the inequalities

$$\begin{aligned}
&[y_3(0) \mp x_{20}]\,T^{-1} \pm TU \geqslant (\leqslant)\, y^{\mp}{}_2 > (<)(\Omega \pm TU)\times \\
&\quad \exp[-x_{20} + \Omega T \pm y_3(0)]\,(T^2 U)^{-1} \\
&[y_3(0) \mp x_{20}]\,T^{-1} \pm TU \geqslant (\leqslant)\, y^{\mp}{}_2 > (<)\, 2TU \mp TU \times \\
&\quad \exp[x_{20} \mp y_3(0)]\,(T^2 U)^{-1}
\end{aligned} \tag{2.7}$$

Here the inequality signs within the parentheses correspond to the lower plus or minus signs.

The initial states $\{y_i(0)\} \in S_3$ are such that it turns out to be possible to find a control, ensuring a nonviolation of the constraints, by proceeding from the necessary conditions of the maximum principle (in the case being examined $\psi_3(t) \not\equiv 0$ and $d\mu / dt \equiv 0$). If $\{y_i(0)\} \overline{\in} S_{1,2,3}$, then such initial conditions belong to the fourth domain S_4 and correspond to the degenerate case. The initial conditions from which trajectories are constructed, passing through the rest point Π_2, belong here.

3. Synthesis of the control. The synthesis consists in determining the membership of the initial conditions in one of the domains S_k listed and in indicating the sequence of sign changes of the control as a function of the phase state. Let us consider the synthesis of the optimal control for a problem without phase constraints. From Fig. 1 it follows that if the initial phase point $\{y_1(0), y^-{}_2(0)\}$ of system (1.1) is located below the curve $y_2 = f_1(y_1)$, then the optimal control is $u^\circ = -U$ from the initial point up to the line $1-2$ with a subsequent switching on it and with a motion along the line $4-5$ up to the final point. If, however, the initial phase point $\{y_1(0), y_2^+(0)\}$ is located above the curve $y_2 = f_2(y_1)$, the optimal control is $u^\circ = +U$ from the initial point up to the line $5-6$ with a subsequent switching on it to $u^\circ = -U$ and with a motion along the line $2-3$ up to the final point. We note that both points $\{y_1(0), y_2^\mp(0)\}$ cannot be simultaneously situated between the lines $y_2 = f_{1(2)}(y_1)$. Consequently, the above-mentioned rule enables us to choose the optimal control uniquely in the problem without phase constraints. We can find the values of the coordinates $y_2[t_p\pm] = g[y_1(0), y_2\pm(0)]$, for which the switching of the control takes place (by solving Eqs.(2.4) and (2.6) simultaneously). The time for passing from the initial point to the point where the switching of the control takes place equals $t_p\pm = T\ln y_2(0)\, y^{-1}_2[t_p\pm]$. We find $y_3[t_p\pm] = y_3(0) \pm TUt_p\pm$.

3.1. If even one of the inequalities

$$
\begin{aligned}
&2TU - TU\exp[x_{20} - y_3(t_p^-)]\ (T^2U)^{-1} \geqslant g[y_1(0), y_2^-(0)] \qquad (3.1)\\
&y_3(t_p^-) < x_{20}\\
&[y_3(t_p^-) - x_{20}]\ T^{-1} + TU \geqslant g[y_1(0), \quad y_2^-(0)]\\
&x_{20} \leqslant y_3(t_p^-) < x_{20} + \Omega T\\
&\Omega + TU > g[y_1(0), y_2^-(0)],\ y_3(t_p^-) > x_{20} + \Omega T\\
&\Omega + TU > g[y_1(0), y_2^-(0)] \geqslant 2TU + (\Omega - TU)\exp[\Omega T -\\
&\quad x_{20} - y_3(t_p^-)]\ (T^2U)^{-1},\ y_{31} \leqslant y_3(t_p^-) \leqslant x_{20} + \Omega T\\
&\Omega + TU > g[y_1(0), \quad y_2^-(0)] \geqslant TU + [y_3(t_p^-) + x_{20}]\ T^{-1},\\
&y_3(t_p^-) < y_{31}\\
&\quad 2TU + TU\exp[x_{20} + y_3(t_p^+)]\ (T^2U)^{-1} \leqslant g[y_1(0), y_2^+(0)]\\
&y_3(t_p^+) \geqslant \Omega T - x_{20}\\
&\Omega - TU > g[y_1(0), y_2^+(0)] \geqslant -2TU + (\Omega + TU)\times\\
&\quad\cdot \exp[y_3(t_p^+) - \Omega T - x_{20}]\ (T^2U)^{-1},\ x_{20} + \Omega T \leqslant y_3(t_p^+) \leqslant y_{32}\\
&\Omega - TU > g[y_1(0), y_2^+(0)],\ y_3(t_p^+) < \Omega T - x_{20}\\
&\Omega - TU > g[y_1(0), y_2^+(0)] \geqslant [y_3(t_p^+) - x_{20}]\ T^{-1} + TU\\
&y_3(t_p^+) > y_{32}
\end{aligned}
$$

is satisfied, then $\{y_i(0)\} \in S_1$ and the synthesis can be effected without going onto the constraints, by the rule indicated above. Here y_{31} and y_{32} are the roots, largest in modulus, of the equations

$$
\begin{aligned}
(\Omega - TU)\exp[x_{20} + \Omega T - y_{31}]\ (T^2U) &= (y_{31} + x_{20})\ T^{-1} - TU\\
(\Omega + TU)\exp(x_{20} + \Omega T - y_{32})\ (T^2U)^{-1} &= (x_{20} - y_{32})T^{-1} + TU
\end{aligned}
$$

3.2. Solutions do not exist for initial conditions belonging to domain S_2, and satisfying inequalities (2.7).

3.3. The initial conditions on the plane $\{y_2, y_3\}$, belonging to S_3, satisfy the equation

$$y_2^-(0) = TU\exp[y_3(0) - x_{20}]\ (T^2U)^{-1}$$

The inequalities

$$(TU - \Omega)\exp[y_1(0) - y_{11}]\ (TU + \Omega)^{-1} \geqslant y_2^-(0) \geqslant TU$$
$$y_{11} \leqslant y_1(0) \leqslant y_{12}, \quad y_{21} \geqslant y_2^-(0) \geqslant TU, \quad y_{12} < y_1(0)$$

where y_{11}, y_{12}, y_{21} are determined from the equations

$$y_{11} = \Omega + (TU - \Omega)\ln TU\,(TU - \Omega)^{-1}, \qquad TU \ln TU\,(y_{21} - 2TU)^{-1} = y_{21} - TU, \quad y_{12} = y_{11} - (TU - \Omega)\ln y_{21}\,(TU)^{-1}$$

must be satisfied on the plane $\{y_1, y_2\}$. The synthesis is effected in the following manner: the conditions in Paragraph 3.3 for membership in domain S_3 are verified; the control at the first stage is always negative: $u^\circ = -U$ until the system reaches the constraint with respect to x_2 with zero velocity (point Π_2); during the time $\Delta t_0 = (y_{22} - y_{11})\,T\Omega^{-1}$, $y_{22} = y_1(0) + (TU + \Omega)\ln TU\,[y_2^-(0)]^{-1}$ the system stays on the constraint with zero control and later it is switched to $u^\circ = +U$ which brings the system to the final state.

3.4. Initial positions, not belonging to the three domains mentioned, belong to S_4 and correspond to the degenerate case. The following arguments can establish the form of the control here. The system's trajectories consist of three segments: going onto the constraint Γ_1 at the rest point Π_2 without falling into the forbidden domain S_2, remaining at point Π_2 for some time and the time-optimal coming off the phase constraint and going into the required final conditions. The total time of motion must be minimal. If t_c is the instant the system comes off the constraint and no longer returns to it, then, proceeding from the requirements of time-optimality and of falling into the required final conditions, the optimal control is $u^\circ = +U$ on the last segment during the time $\Delta t_f = t_f - t_c$ (we have in mind the motion from Π_2 to the line $\Omega - TU$). The relations

$$x_3(t_f) = \Omega, \quad x_3(t_c) = 0, \quad \Delta t_f = T\ln TU\,(TU - \Omega)^{-1}$$

are valid.

The value $x_2(t_f)$ is determined from the condition $x_2(t_c) = x_{20}$

$$x_2(t_f) = x_{20} + T^2 U \ln TU\,(TU - \Omega)^{-1} + T\,(TU - \Omega) \tag{3.2}$$

With due regard to (1.1) and (3.2) we get that the maximum time-optimal duration is $t_f = [x_2(t_f) - x_1(0) - x_{20}]\Omega^{-1}$. By the instant t_c the control must bring the object to the state

$$x_1(t_c) = x_2(t_f) - x_{20} - \Omega\Delta t_f, \quad x_2(t_c) = x_{20}, \quad x_3(t_c) = 0 \tag{3.3}$$

On the plane $\{y_2, y_3\}$ this state is located on Γ_1 with the coordinates $\{x_{20}, \Omega - TU\}$. On the interval $0 \leqslant \tau \leqslant t_c$ the control must be such that the system does not fall into the forbidden domain S_2

$$\{y_i(\tau), \quad 0 \leqslant \tau \leqslant t_c\} \cap S_2 = 0 \tag{3.4}$$

The optimal control on the segment $[0, t_c]$, necessarily satisfying conditions (3.3) and (3.4), is not unique.

In fact, the relation $t_c - t_m = \Delta t_b \geqslant 0$ must be satisfied for the minimal time t_m of translation of the point $y_i(0) \in S_4$ onto boundary Γ_1 at the point $\{x_{20}, \Omega - TU\}$. Obviously, infinite number of controls exist in the time interval Δt_b, translating the object from the phase point $y_i(t_m)$ to $y_i(t_c)$. By virtue of this there exists a set of controls translating the object from S_4 to the final point.

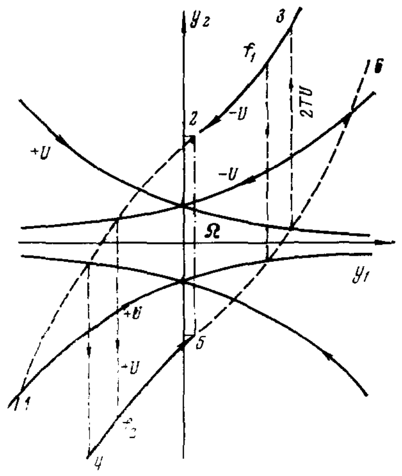

Fig. 1

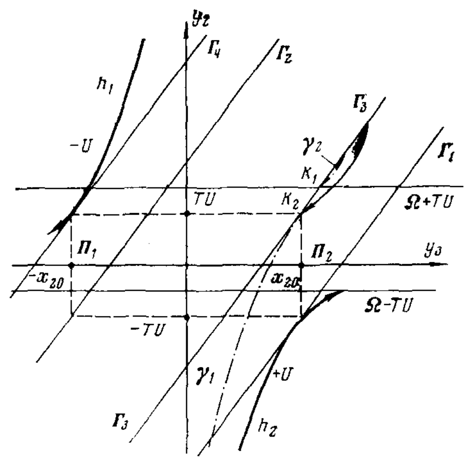

Fig. 2

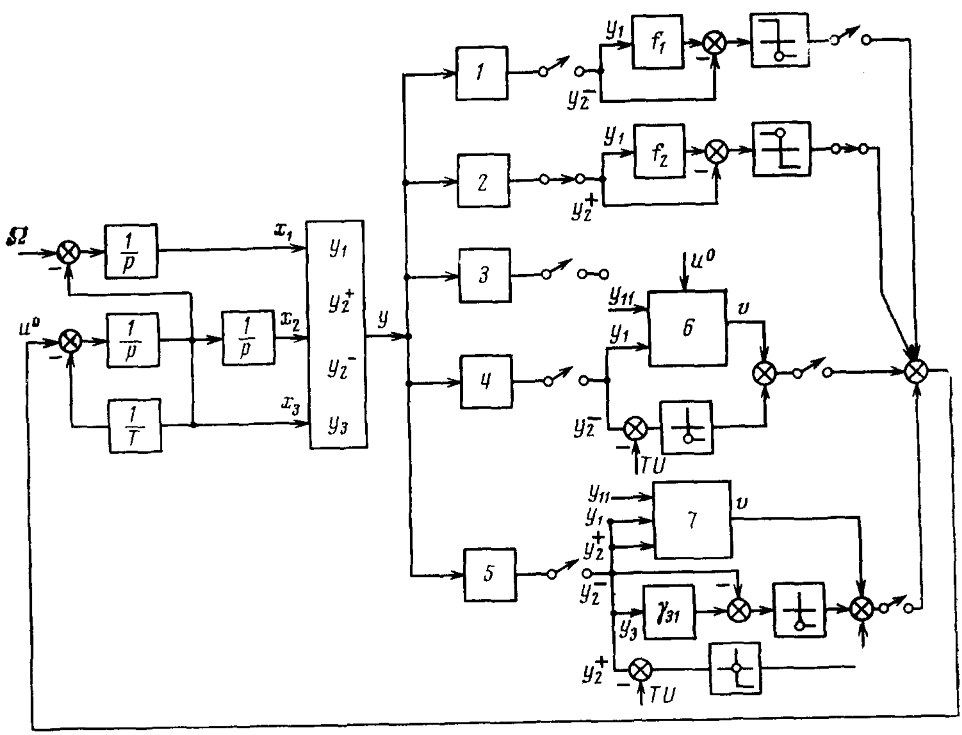

Fig. 3

Therefore, in practice, the synthesis can be effected as follows. The system is translated from the initial point to a rest point with one switching of the control; the system remains on the constraint during the time $\Delta t_1 = [x_1(t_b) - x_1(t_1)]\ \Omega^{-1}$; later, the control is switched to $u^\circ = +U$ which translates the object to the final state. Figure 3 shows the block diagram of the optimal control synthesis circuit.

Control in the closed loop is effected in the following way. The membership of the initial conditions to domains S_k, $k = 1-3$, is verified in blocks $1-4$ by formulas (3.1), (2.7) and the formulas in Paragraph 3.3, respectively. If $\{y_i(0)\} \overline{\in} S_{1,2,3}$, then the initial conditions belong to S_4 (block 5). When the membership conditions have been satisfied, the appropriate switches are closed, which transfer the circuit into a closed-loop state (the remaining switches are open). For S_1 the control is carried out along the switching line; for S_3 and S_4 the control signal is formed in blocks 6 and 7 where the value 0 or U is conferred on the quantity v when any of the following conditions is satisfied:

for block 6

a) $U \operatorname{sign} u^\circ(t) - \Omega T^{-1} < 0,\ y_{11} - y_1 < 0;\ \ v = 0$; (b) $y_{11} - y_1 = 0;\ v = U$; (c) $U \operatorname{sign} u^\circ(t) - \Omega T^{-1} > 0,\ y_{11} - y_1 < 0;\ v = U$

for block 7

a) $y_2^+ \leqslant TU,\ y_{11} - y_1 < 0;\ v = 0$; (b) $y_{11} - y_1 = 0;\ v = U$; c) $y_2^+ > -TU,\ y_{11} - y_1 < 0,\ v = U$

4. Solution for arbitrary Ω, TU, x_{20}. We see that when $\Omega < 0, TU > |\Omega|$ and $x_{20} > T^2U$ the realization of the rest point with a subsequent satisfaction of the right-end conditions is possible if the coordinate of the rest point with respect to y_3 takes a negative value. The whole structure on the plane $\{y_2, y_3\}$ becomes symmetric relative to the origin. The case $TU \leqslant |\Omega|$ $(\Omega > 0, \Omega < 0)$ has no physical meaning since under such a condition the coordinate $x_3(t_f)$ cannot take the value Ω. If $TU \leqslant \Omega = 0$, then the problem does not have a solution when $x_2(0) + x_1(0) < x_{20}$; the analysis in Paragraphs 3.1 and 3.2 remains valid in other respects. The mutual position of the strips (2.5) is determined by the relation of the quantities x_{20} and T^2U. If $x_{20} \geqslant T^2U$, the strips have a common part. The case $x_{20} < T^2U$ (the strips do not have common parts) can be considered similarly.

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